THE PLANE PROBLEM OF COUPLE STRESS THEORY OF ELASTICITY FOR AN INFINITE PLANE WEAKENED BY A FINITE NUMBER OF CIRCULAR HOLES

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Only problems on the stress concentration near a single round hole have been investigated [1 to 3] in the context of the plane problem of the theory of elasticity with couple stress.

In the present paper we offer a method for solving plane problems of theory of elasticity with couple stress for a plane, weakened by a finite number of arbitrarily situated circular unconnected holes. It is shown that the problems are reducible to infinite systems of algebraic equations. The basic inequalities and relations required to prove the quasi-regularity of infinite systems and the uniqueness of the solution are obtained.

The problem of two circular holes of equal size is discussed in detail. The quasi-regularity and uniqueness of the solution of the resulting infinite system of algebraic equations is proved under the following conditions: (1) a self-balanced load is applied to the contours; (2) the normal and tangential components are continuous functions whose first derivatives satisfy the Dirichlet condition; (3) the distributed couples and first derivatives are continuous functions, while the second derivative satisfies the Dirichlet condition.

Following [1], let us write out the basic equations of plane deformation of theory of elasticity with couple stresses in the polar coordinate system.*

The basic equations are

$$\Delta\Delta\varphi = 0, \qquad \Delta\left(\Delta - \frac{r_0^2}{l^2}\right)\psi = 0 \tag{1}$$

The compatibility equations are

$$\frac{\partial}{\partial r}(r_0^2 - l^2\Delta)\psi = -2(1-\nu)l^2\frac{1}{r}\frac{\partial}{\partial\theta}\Delta\varphi \tag{2}$$

* All coordinates and linear dimensions are dimensionless and referred to r_0.

$$\frac{1}{r}\frac{\partial}{\partial\theta}(r_0^2 - l^2\Delta)\psi = 2(1-\nu)\,l^2\frac{\partial}{\partial r}\Delta\varphi$$

The stresses can be detemined from the relations

$$\sigma_r = \frac{1}{r_0^2}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi - \frac{1}{r_0^2}\frac{\partial^2}{\partial r\,\partial\theta}\frac{1}{r}\psi,$$

$$\sigma_\theta = \frac{1}{r_0^2}\frac{\partial^2}{\partial r^2}\varphi + \frac{1}{r_0^2}\frac{\partial^2}{\partial r\,\partial\theta}\frac{1}{r}\psi$$

$$\tau_{r\theta} = -\frac{1}{r_0^2}\frac{\partial^2}{\partial r\,\partial\theta}\frac{1}{r}\varphi - \frac{1}{r_0^2}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi \tag{3}$$

$$\tau_{\theta r} = -\frac{1}{r_0^2}\frac{\partial^2}{\partial r\,\partial\theta}\frac{1}{r}\varphi + \frac{1}{r_0^2}\frac{\partial^2}{\partial r^2}\psi$$

$$\mu_r = \frac{1}{r_0}\frac{\partial\psi}{\partial r}, \qquad \mu_\theta = \frac{1}{r_0^2}\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

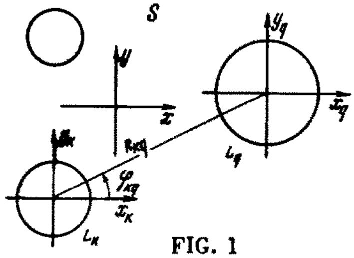

FIG. 1

Let us consider the state of stress of an infinite plane weakened by m circular holes (Fig. 1) of radius $r_0 R_k$ $(k = 1, \ldots, m)$.

To each of the holes we assign a coordinate system (x_k, y_k),

$$\begin{aligned} x + iy = z, \quad x_k + iy_k = z_k, \quad z_k = r_k e^{i\theta_k}, \quad z = re^{i\theta} \\ z = z_k + l_k, \quad z_k = z_q + R_{kq}e^{i\varphi_{kq}} \qquad (k, q = 1, \ldots, m) \end{aligned} \tag{4}$$

The solution of the problem then reduces to the solution of Equations (1) under condition (2) for the m-connected region S bounded by the contour $L = L_1 + \ldots + L_2$. It should be noted that this solution must satisfy the conditions of single-valued displacements and the conditions 'at infinity'; the latter reduce to the decay of the stress and strain components.

The solution of Equations (1) under condition (2) for an m-connected region can be written as

$$\varphi = \sum_{q=1}^{m} A_0^{(q)} \ln r_q + \sum_{q=1}^{m}\sum_{p=1}^{\infty}\left[\begin{matrix} A_p^{(q)} \\ B_p^{(q)} \end{matrix} r_q^{-p} + \begin{matrix} C_p^{(q)} \\ D_p^{(q)} \end{matrix} r_q^{-p+2}\right]\begin{matrix}\cos \\ \sin\end{matrix}\, p\theta_q \tag{5}$$

$$\psi = \sum_{q=1}^{m} F_0^{(q)} K\left(r_q\frac{r_0}{l}\right) + \sum_{q=1}^{m}\sum_{p=1}^{\infty} \begin{matrix} F_p^{(q)} \\ B_p^{(q)} \end{matrix} K_p\left(r_q\frac{r_0}{l}\right)\begin{matrix}\cos \\ \sin\end{matrix}\, p\theta_q \mp 8(1-\nu)\,l^2 \times$$

$$\times \sum_{q=1}^{m}\sum_{p=1}^{\infty}(p-1)\begin{matrix} D_p^{(q)} \\ C_p^{(q)} \end{matrix} r_q^{-p}\begin{matrix}\cos \\ \sin\end{matrix}\, p\theta_q \tag{6}$$

When the principal vector and principal moment of the stresses applied to each contour are equal to zero, the conditions of single-valued displacements and the conditions 'at infinity' are fulfilled provided we set

$$C_1^{(q)} \equiv D_1^{(q)} = 0 \qquad (q = 1, \ldots, m) \tag{7}$$

Let us represent solution (5) and (6) in the k-th coordinate system. We begin by recalculating the harmonic and biharmonic functions. To this end we consider a function which is analytic for $|z| < \alpha|$ and expand it into a Taylor series for $|z| < |\alpha|$,

$$f(z, p, \alpha) = (\alpha - z)^{-p}, (\alpha - z)^{-p} = \sum_{n=0}^{\infty} z^n \frac{(p+n-1)!}{n!(p-1)!} \frac{1}{\alpha^{p+n}} \tag{8}$$

Setting here $\alpha = R_{kq} e^{i\varphi_{kq}}$ and $z = z_k$ (4), we obtain

$$r_q^{-p} \begin{matrix}\cos\\ \sin\end{matrix} p\theta_q = \pm \sum_{n=0}^{\infty} (-1)^p \frac{(p+n-1)!}{n!(p-1)!} \frac{r_k^n}{R_{kq}^{p+n}} \left[\cos n\theta_k \begin{matrix}\cos\\ \sin\end{matrix} (n+p)\varphi_{kq} + \right.$$
$$\left. + \sin n\theta_k \begin{matrix}\sin\\ \cos\end{matrix} (n+p)\varphi_{kq}\right] \tag{9}$$

$$r_p^{-p+2} \begin{matrix}\cos\\ \sin\end{matrix} p\theta_q = \pm \sum_{n=0}^{\infty} (-1)^p \frac{(p+n-1)!}{n!(p-2)!} \frac{1}{R^{p+n-2}} \left[\frac{r_k^n}{p+n-1} - \frac{1}{R_{kq}^2} \frac{r_k^{n+2}}{n+1}\right] \times$$
$$\times \left[\cos n\theta_k \begin{matrix}\cos\\ \sin\end{matrix} (n+p)\varphi_{kq} + \sin n\theta_k \begin{matrix}\sin\\ \cos\end{matrix} (n+p)\varphi_{kq}\right] \pm (-1)^{p+1} \frac{r_k}{R_{kq}^{p-1}} \times \tag{10}$$
$$\times \left[\cos\theta_k \begin{matrix}\cos\\ \sin\end{matrix} (p-1)\varphi_{kq} + \sin\theta_k \begin{matrix}\sin\\ \cos\end{matrix} (p-1)\varphi_{kq}\right] \qquad (r_k < R_{kq})$$

For $\ln r_q$ we have the following series [4]:

$$\ln r_q = \ln R_{kq} - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r_k}{R_{kq}}\right)^n (\cos n\theta_k \cos n\varphi_{kq} + \sin n\theta_k \sin n\varphi_{kq}) \qquad (r_k < R_{kq}) \tag{11}$$

From the addition theorem for cylindrical functions [5] we obtain

$$K_p(cr_q) \begin{matrix}\cos\\ \sin\end{matrix} p\theta_q = (-1)^p \sum_{n=0}^{\infty} \varepsilon_n I_n(cr_k) \left\{\left[K_{p+n}(cR_{kq}) \begin{matrix}\cos\\ \sin\end{matrix} (n+p)\varphi_{kq} \pm \right.\right.$$
$$\left. \pm K_{p-n}(cR_{kq}) \begin{matrix}\cos\\ \sin\end{matrix} (n-p)\varphi_{kq}\right] \cos n\theta_k \pm \left[K_{p+n}(cR_{kq}) \begin{matrix}\sin\\ \cos\end{matrix} (n+p)\varphi_{kq} \pm \right. \tag{12}$$
$$\left.\left. \pm K_{p-n}(cR_{kq}) \begin{matrix}\sin\\ \cos\end{matrix} \varphi_{kq}\right] \sin n\theta_k\right\}, \qquad \varepsilon_n = \begin{cases} 0.5 & n = 0 \\ 1 & n > 0 \end{cases}; \ c = \text{const}; \ r_k < R_{kq}$$

Substituting (9) to (12) into (5) and (7), we obtain the solution in the k-th coordinate system in the form of Fourier series, which enable us to satisfy the boundary conditions at the k-th contour ($k = 1, \ldots, m$).

Let us introduce into (5) and (6) the following new constants, the need for which will become more obvious below:

$$\begin{aligned} A_p^{(q)} &= x_{p,1}^{(q)}, & C_p^{(q)} &= x_{p,2}^{(q)}, & E_p^{(q)} &= I_p\left(R_q \frac{r_0}{l}\right) x_{p,3}^{(q)} \\ B_p^{(q)} &= x_{p,4}^{(q)}, & D_p^{(q)} &= x_{p,5}^{(q)}, & F_p^{(q)} &= I_p\left(R_q \frac{r_0}{l}\right) x_{p,6}^{(q)} \end{aligned} \tag{13}$$

Substituting (5) and (6) into the boundary conditions at the k-th contour and taking account of (13) and (9) to (13), we obtain the infinite system in the form

$$B_n^{(k)*} X_n^{(k)} + \sum_{q=1}^{m}{}' \sum_{p=0}^{\infty} B_{n,p}^{(k,q)} X_p^{(q)} = B_n^{(k)**} \qquad \begin{pmatrix} k = 1, \ldots, m \\ n = 0, \ldots \infty \end{pmatrix} \tag{14}$$

Here and below the prime next to the summation sign means that the term with $q = k$ is omitted in the sum. $X_n^{(k)}$ and $B_n^{(k)**}$ are six-dimensional vector columns; $B_n^{(k)*}$ and $B_{n,p}^{(k,q)}$ are the six-dimensional matrices

$$X_n^{(k)} = \{x_{n,j}^{(k)}\}, \qquad B_n^{(k)**} = \{b_j^{**}(n, k)\}, \qquad B_n^{(k)*} = \| b_{ij}^{*}(n, k) \| \\ B_{n,p}^{(k,q)} = \| b_{ij}(n, p, k, q) \| \qquad (i, j = 1, \ldots, 6) \tag{15}$$

Multiplying (14) by $1/B_n^{(k)*}$ (it can be shown that $B_n^{(k)*}$ is a nondegenerate matrix), we obtain the infinite system in the canonical form [6],

$$X_n^{(k)} + \sum_{q=1}^{m}{}' \sum_{p=0}^{\infty} A_{n,p}^{(k,q)} X_p^{(q)} = b_n^{(k)} \qquad \begin{pmatrix} k = 1, \ldots, m \\ n = 0, \ldots, \infty \end{pmatrix} \\ A_{n,p}^{(k,q)} = \| a_{ij}(n, p, k, q) \|, \qquad b_n^{(k)} = \{b_j(n, k)\} \qquad (i, j = 1, \ldots, 6) \tag{16}$$

Let us cite certain relations required for proving the quasiregularity and uniqueness of the solutions of infinite systems of the form (16). From [7] and Stirling's formula for $|n| \gg |z|$, we have

$$K_n(z) \sim \frac{(n-1)!}{2}\left(\frac{2}{z}\right)^n, \qquad I_n(z) \sim \frac{1}{n!}\left(\frac{z}{2}\right)^n \tag{17}$$

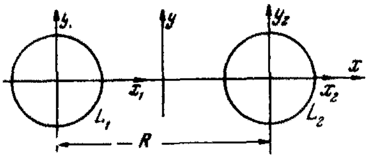

FIG. 2

In addition, for any n we can obtain [5] the estimate

$$|I_n(z)| < K_1 \frac{1}{n!}\left(\frac{z}{2}\right)^n, \; K_1, \; K_2, \ldots = \text{const} \tag{18}$$

From (17) and (18), for a large n, we obtain the following estimate:

$$\left| I_p\left(R_q \frac{r_0}{l}\right) \frac{K_{p+n}(R_{kq} r_0/l)\cos(n+p)\varphi_{kq} \pm K_{p-n}(R_{kq} r_0/l)\cos(n-p)\varphi_{kq}}{K_n(R_q r_0/l)} \right| < \\ < K_2 \frac{(p+n)!}{n!\,p!}\left(\frac{R_q}{R_{kq}}\right)^{p+n} \tag{19}$$

Relations (17) to (19) and various implications of expansion (8) are sufficient to prove the quasiregularity and uniqueness of the solutions of infinite systems under boundary conditions of fairly general form in the case of a finite number of arbitrarily situated nonadjacent holes. For conciseness, let us consider two holes of equal size.

Let us examine the state of stress of an infinite plate weakened by two equal circular holes (Fig. 2) of radius r_0, their centers separated by the distance Rr_0,

$$R_k = 1, \quad R_{kq} = R, \quad \varphi_{12} = 0, \quad \varphi_{21} = \pi \qquad (k = 1, 2) \tag{20}$$

We shall assume that a load symmetrical with respect to the x- and y-axes has been applied to the hole contours. The boundary conditions can then be written as

$$\sigma_{r_1}\big|_{r_1=1} = \sum_{n=0}^{\infty} \sigma_r^{(n)} \cos n\theta_1, \qquad \tau_{r_1\theta_1}\big|_{r_2=1} = \sum_{n=1}^{\infty} \tau_{r\theta}^{(n)} \sin n\theta_1 \\ \mu_{r_1}\big|_{r_1=1} = \sum_{n=1}^{\infty} \mu_r^{(n)} \sin n\,\theta_1, \qquad \sigma_{r_2}\big|_{r_2=1} = \sum_{n=0}^{\infty} (-1)^n \sigma_r^{(n)} \cos n\theta_2 \tag{21}$$

$$\tau_{r_2\theta_2}|_{r_2=1} = \sum_{n=1}^{\infty} (-1)^n \tau_{r\theta}^{(n)} \cos n\theta_2, \qquad \mu_{r_2}|_{r_2=1} \sum_{n=1}^{\infty} (-1)^n \mu_r^{(n)} \sin n\theta_2$$

In accordance with (5) and (6) the solution satisfying the conditions of traction and geometric symmetry are taken in the form

$$\varphi = A(\ln r_1 + \ln r_2) + \sum_{n=1}^{\infty} \left\{ A_n \left[\frac{\cos n\theta_1}{r_1^n} + \frac{(-1)^n \cos n\theta_2}{r_2^n} \right] + C_n \left[\frac{\cos n\theta_1}{r_1^{n-2}} + \frac{(-1)^n \cos n\theta_2}{r_2^{n-2}} \right] \right\} \tag{22}$$

$$\psi = \sum_{n=1}^{\infty} E_n \left[K_n\left(r_1 \frac{r_0}{l}\right) \sin n\theta_1 + (-1)^n K_n\left(r_2 \frac{r_0}{l}\right) \sin n\theta_2 \right] + 8(1-\nu)\frac{l^2}{r_0^2} \sum_{n=1}^{\infty} C_n(n-1) \times \left[\frac{\sin n\theta_1}{r_1^n} + \frac{(-1)^n \sin n\,\theta_2}{r_2^n} \right] \tag{23}$$

$$A_n = A_n^{(1)}, \quad C_n = C_n^{(1)}, \quad E_n = E_n^{(1)} \tag{24}$$

In addition, we take account of the relations

$$A_n^{(2)} = (-1)^n A_n^{(1)}, \qquad C_n^{(2)} = (-1)^n C_n^{(1)}, \qquad E_n^{(2)} = (-1)^n E_n^{(1)} \tag{25}$$

which are valid by virtue of the geometric and traction symmetry of the problem. A solution in the form (22) and (23) enables us to satisfy boundary conditions only at the contour of the left-hand hole; the boundary conditions at the contour of the right-hand hole are then satisfied automatically. From the conditions that the principal vector and principal moment at each contour are equal to zero and from the conditions 'at infinity', we obtain

$$\sigma_r^{(1)} = \tau_{r\theta}^{(1)}, \qquad c_1 = 0 \tag{26}$$

In accordance with (13), we introduce the following new constants into (22) and (23):

$$A_n = x_{n,1}, \qquad C_n = x_{n,2}, \qquad E_n = I_n\left(\frac{r_0}{l}\right) x_{n,3} \tag{27}$$

Substituting (22) and (23) into the boundary conditions at the left-hand hole (21) and taking account of (3), (9) to (12), and (27), we obtain the infinite system. Let us consider this system for $n = 0$ and define

$$A = r_0^2 \left[\sigma_r^{(0)} + 2 \sum_{p=1}^{\infty} x_{p,1} \frac{p-1}{r_0^2 R^{p-2}} \right] \tag{28}$$

Taking account of (28), we obtain an infinite system of the form

$$B_n^* X_n + \sum_{p=1}^{\infty} B_{n,p} X_p = B_n^{**} \qquad (n = 1, \ldots, \infty) \tag{29}$$

$$X_n = \{x_{n,j}\}, \quad B_n^{**} = \{b_j^{**}(n)\}, \quad B_n^* = \| b_{ij}^*(n) \|, \quad B_{n,p} = \| b_{ij}(n, p) \quad (i, j = 1, 2, 3)$$

Here for $n = 1$ the first and second equations coincide and $x_{1,2} = 0$ by virtue of (27).

Let us cite the values of $b_{ij}^*(n)$, $b_{ij}(n, p)$, and $b_j^{**}(n)$ for $i, j = 1, 2, 3$:

$$b_{11}^*(n) = -\frac{n(n+1)}{r_0^2}, \quad b_{12}^*(n) = -\frac{n^2+n-2}{r_0^2} + 8(1-\nu)\frac{l^2}{r_0^4} n(n^2-1)$$

$$b_{13}^*(n) = -\frac{n}{r_0^2} I_n\left(\frac{r_0}{l}\right)\left[\frac{r_0}{l} K_n'\left(\frac{r_0}{l}\right) - K_n\left(\frac{r_0}{l}\right)\right] \quad b_{21}^*(n) = -\frac{n(n+1)}{r_0^2}$$

$$b_{22}^*(n) = -\frac{n(n-1)}{r_0^2} + 8(1-\nu)\frac{l^2}{r_0^4} n(n^2-1)$$

$$b_{23}^*(n) = -\frac{1}{r_0^2} I_n\left(\frac{r_0}{l}\right)\left[\frac{r_0}{l} K_n'\left(\frac{r_0}{l}\right) - n^2 K_n\left(\frac{r_0}{l}\right)\right]$$

$$b_{31}^*(n) = 0, \quad b_{32}^*(n) = -8(1-\nu)\frac{l^2}{r_0^4} n(n-1), \quad b_{33}^*(n) = \frac{1}{l} K_n'\left(\frac{r_0}{l}\right) I_n\left(\frac{r_0}{l}\right)$$

$$b_{11}(n, p) = -\frac{(p+n-1)!}{(n-2)!(p-1)!}\frac{1}{r_0^2 R^{p+n}},$$

$$b_{12}(n, p) = \frac{1}{r_0^2 R^{p+n-2}}\frac{(p+n-1)!}{n!(p-2)!}\left[\frac{n^2-n-2}{R^2} - \frac{n(n-1)}{p+n-1}\right] +$$

$$+ 8(1-\nu)\frac{l^2}{r_0^4}\frac{(p+n-1)!}{(n-2)!(p-2)!}\frac{1}{R^{p+n}} + 2\frac{(p-1)(n-1)}{r_0^2 R^{p+n-2}}$$

$$b_{13}(n, p) = \frac{n}{r_0^2} I_p\left(\frac{r_0}{l}\right)\left[K_{p+n}\left(R\frac{r_0}{l}\right) - K_{p-n}\left(R\frac{r_0}{l}\right)\right]\left[\frac{r_0}{l} I_n'\left(\frac{r_0}{l}\right) - I_n\left(\frac{r_0}{l}\right)\right]$$

$$b_{21}(n, p) = \frac{1}{r_0^2}\frac{(p+n-1)!}{(n-2)!(p-1)!}\frac{1}{R^{p+n}}$$

$$b_{22}(n, p) = \frac{1}{r_0^2}\frac{(p+n-1)!}{(n-1)!(p-2)!}\frac{1}{R^{p+n-2}}\left(\frac{n-1}{p+n-1} - \frac{1}{R^2}\right) + \tag{30}$$

$$+ 2\frac{(n-1)(p-1)}{r_0^2 R^{p+n-2}} - 8(1-\nu)\frac{l^2}{r_0^4}\frac{(p+n-1)!}{(n-2)!(p-1)!}\frac{1}{R^{p+n}}$$

$$b_{23}(n, p) = \frac{1}{r_0^2} I_p\left(\frac{r_0}{l}\right)\left[K_{p+n}\left(R\frac{r_0}{l}\right) - K_{p-n}\left(R\frac{r_0}{l}\right)\right] \times$$

$$\times\left[\frac{r_0}{l} I_n'\left(\frac{r_0}{l}\right) - n^2 I_n\left(\frac{r_0}{l}\right)\right]$$

$$b_{31}(n, p) = 0, \quad b_{32}(n, p) = -8(1-\nu)\frac{l^2}{r_0^3}\frac{(p+n-1)!}{(n-1)!(p-2)!}\frac{1}{R^{p+n}}$$

$$b_{33}(n, p) = -\frac{1}{l} I_p\left(\frac{r_0}{l}\right)\left[K_{p+n}\left(R\frac{r_0}{l}\right) - K_{p-n}\left(R\frac{r_0}{l}\right)\right] I_n'\left(\frac{r_0}{l}\right)$$

$$b_1^{**}(n) = \sigma_r^{(n)} - \sigma_r^{(0)}\frac{n-1}{R^n}, \quad b_2^{**}(n) = \tau_{r\theta}^{(n)} + \sigma_r^{(0)}\frac{n-1}{R^n}, \quad b_3^{**}(n) = \mu_r^{(n)}$$

Consider now the determinant $|B_n^*|$. Making use of asymptotic forms more exact as compared with (17), for large n we obtain

$$|B_n^*| \sim 2\frac{3-2\nu}{r_0^5} n^3 \tag{31}$$

From (31) it follows that the matrix B_n^* is nondegenerate. Multuplying (29) by $1/B_n^*$, we obtain the infinite system in canonical form,

$$X_n + \sum_{p=1}^{\infty} A_{n,p} X_p = B_n \qquad (n = 1, \ldots, \infty) \tag{32}$$

For $n = 1$ the first and second equations in (32) coincide, and $x_{1,2} = 0$;

$$A_{n,p}=\frac{B_{n,p}}{B_n{}^*}, \quad B_n=\frac{B_n{}^{**}}{B_n{}^*}, \quad A_{n,p}=\|a_{ij}(n,p)\|, \quad B_n=\{b_j(n)\} \qquad (i,j=1,2,3) \tag{33}$$

From (17) to (19), (30), (31), and (33) we obtain the estimate

$$|a_{ij}(n,p)|<K_3\frac{(p+n)!}{(n-3)!\,(p-1)!}\frac{1}{R^{p+n}} \tag{34}$$

We shall assume that $\sigma_{r1}|_{r1}=1$ and $\tau_{r1\,\theta\,1}|_{r1}=1$ are continuous functions whose first derivatives satisfy the Dirichlet condition; $\mu_{r1\,r1}=1$ and its first derivative are continuous, its second derivative satisfying the Dirichlet condition. In this case from (17), (18), (30), (31), (33), and (8) we obtain

$$|b_j(n)|<\frac{K_3}{n} \tag{35}$$

Let us consider the sum

$$\sum_{p=1}^{\infty}|a_{ij}(n,p)| \tag{36}$$

It is clear that these sums are bounded for any n. Let us investigate the sum for large n. From (34) we obtain

$$\sum_{p=1}^{\infty}|a_{ij}(n,p)|<K_3\sum_{p=1}^{\infty}\frac{(p+n)!}{(n-3)!\,(p-1)!}\frac{1}{R^{p+n}} \tag{37}$$

In order to sum the series on the right-hand side of (37), let us consider the function

$$\frac{d}{dz}\alpha n(n-1)(n-2)f(z,n+1,\alpha) \tag{38}$$

Substituting (8) into (38) and differentiating term by term, we obtain

$$\frac{\alpha n(n^2-1)(n-2)}{(\alpha-z)^{n+2}}=\sum_{p=1}^{\infty}z^{p-1}\frac{(p+n)!}{(n-3)!\,(p-1)!}\frac{1}{\alpha^{p+n}} \qquad (|z|<|\alpha|) \tag{39}$$

Setting $z=1$ and $\alpha=R$ in (38), we find from (37) and (38) that

$$\sum_{p=1}^{\infty}|a_{i,j}(n,p)|<K_4\frac{n(n^2-1)(n-2)}{(R-1)^{n+2}} \qquad (i,j=1,2,3) \tag{40}$$

In the case of nonadjacent holes, $R>2$. This implies that

$$\sum_{p=1}^{\infty}|a_{ij}(n,p)|\underset{n\to\infty}{\to}0 \qquad (i,j=1,2,3) \tag{41}$$

i.e. that there is an n^0 such that

$$\sum_{p=1}^{\infty}|a_{ij}(n,p)|<1 \qquad (n=n^0+1,\ldots,\infty) \tag{42}$$

From (35) and (42) it follows that there exists a $K>0$ such that

$$|b_j(n)| \leqslant K\left(1 - \sum_{p=n^\circ+1}^{\infty} |a_{ij}(n, p)|\right) \qquad \left(\begin{matrix} n = n^\circ + 1, \ldots, \infty \\ i, j = 1, 2, 3. \end{matrix}\right) \tag{43}$$

Inequalities (42) and (43) indicate that the infinite system (32) is quasiregular and that its solution can therefore be determined by the reduction method.

Let us investigate the uniqueness of the solution of indinite system (32); we consider the series

$$\sum_{p=1}^{\infty} \sum_{n=1}^{\infty} a_{ij}^2 (n, p) \tag{44}$$

The convergence of series (44) follows from (40), convergence of the series

$$\sum_{n=1}^{\infty} n^{k_1} \alpha^{-n+k_2} \qquad (k_1, k_2, \alpha = \text{const}; \ \alpha > 1)$$

and convergence of the double series provided the recurrent series converge in the case of positive terms. The right-hand sides of (32) belong to the space l_2 by virtue of (35). This guarantees the fulfillment of the conditions of applicability of Hilbert's theorem [6] whence it follows that either (1) infinite system (32) has a unique solution belonging to l_2, or, (2) homogeneous system (32) has a solution different from zero. The second case of Hilbert's theorem does not apply, since this would violate the theorem on the uniqueness of the solutions of boundary value problems of couple stress elasticity theory. It is therefore the first case which applies.

In the case of two equal nonadjacent circular holes, the infinite system is quasiregular and has a unique solution which can be determined by the reduction method if (1) $\sigma_{r_1}|_{r_1=1}$ and $\tau_{r_1\theta_1}|_{r_1=1}$ are continuous functions whose first derivatives satisfy the Dirchlet condition, and (2) $\mu_{r_1|r_1=1}$ is a function which is continuous together with its first derivative and whose second derivative satisfies the Dirichlet condition.

We note that the suggested approach can be applied to the solution of a number of other plane problems of the theory of elasticity with couple stress.

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